JOURNAL OF MATERIALS SCIENCE 23 (1988) 928–933

Statistical fibre failure and single crack behaviour in uniaxially reinforced ceramic composites

MUZAFFER SUTCU
General Electric Company, Corporate Research and Development, Schenectady, New York 12301, USA

A lower bound to the work of pull-out is estimated for a ceramic composite under uniaxial loading assuming that matrix crumbling does not occur. Fibre failure is assumed to be governed by Weibull distribution. In order to compute a lower bound to the energy dissipation it is assumed that the failure occurs by a single matrix crack. The fibre/matrix interface is assumed to be constrained by friction only. The work of pull-out estimated from the present model is compared to the energy dissipation before fibre failure which was computed by Aveston, Cooper and Kelly (ACK) in 1971. Comparisons are made with the surface energy of monolithic materials.

1. Introduction

Perhaps the most important characteristic of practical engineering materials is their ability to deform and dissipate energy under severe operating conditions. Ceramic composites which are intended for high-temperature structural applications are made of brittle fibre and matrix material combinations which consume very little energy when failed individually. The composites of these brittle materials must be designed such that the fibres and the matrix phases fail at different locations and at different times with respect to each other, thus allowing relative sliding to occur at the fibre matrix interface. One method of controlling the failure mode of the composite [1, 2] is to introduce chopped or regularly spaced predamaged fibres into the composite and thus control the fibre pull-out length.

For a composite which is unidirectionally reinforced with continuous long fibres, frictional energy dissipation occurs before as well as after fibre failure. Frictional energy dissipation before fibre failure accompanies matrix cracking and is due to elastic stress relaxation. The magnitude of this dissipation is of the order of the surface energy, as will be shown later. Considerably more energy is consumed after fibre failure due to the fact that the fibres slide over long distances inside the matrix. However, because of the large displacements involved this mode of energy dissipation may not always be useful.

The situation can be described by the sketch given in Fig. 1. A tensile specimen is loaded in the fibre direction as shown in Fig. 1a and the apparent stress is plotted against cross-head displacement in Fig. 1b. Apparent stress is defined as the external load divided by the original composite cross-sectional area. In the initial linear elastic region the fibres and the matrix are strained together. Frictional dissipation is absent because there is no relative sliding in this region.

At a certain stress, σ_{cr}, the matrix is multiply cracked at constant stress and an amount of energy, W_d, is dissipated due to the relative sliding at the fibre/matrix interface. It will be shown later that the strain range of this region is comparable to the elastic strain range. Multiple cracking occurs throughout the volume of the body. Therefore, W_d is a volume quantity, rather than surface.

It can be shown that the stress distribution in the matrix does not change after full multiple cracking with additional increase in the external load. The fibres carry all of the additional external load and start failing in a random fashion throughout the volume of the specimen. Eventually, one crack becomes dominant and final separation occurs along this crack. The energy dissipation which occurs after fibre failure is denoted by W_p and is proportional to cross-sectional area. Hence, it may be considered to be a surface quantity.

In this paper a lower bound to W_p is given by assuming that the pull-out process occurs due to one matrix crack only. The case of multiple cracks is treated by Sutcu [3]. The lower bound which is given in this paper is a good measure of toughness for material design purposes.

2. Energy dissipation before fibre failure, W_d

Energy dissipation before fibre failure has been computed by Aveston *et al.* [4] (ACK) assuming that the composite has a pre-existing large crack in the matrix phase and that steady state matrix crack extension takes place whenever the external work done by the surface tractions is equal to the summation of the following three energy terms:

1. surface energy of the matrix;
2. frictional dissipation along the fibre matrix interface on both sides of the crack surface;

0022–2461/88 $03.00 + .12

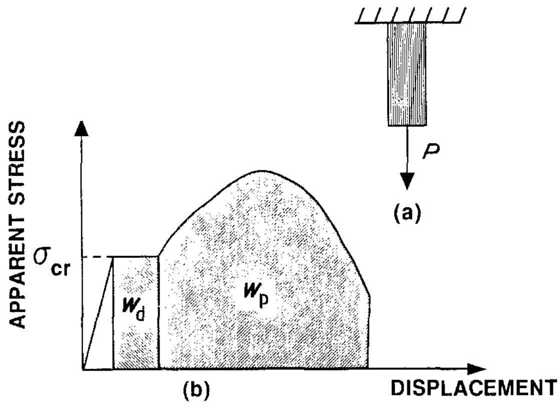

Figure 1 (a) Uniaxial specimen. (b) Typical stress–strain curve.

3. the increase in the elastic energy stored in the crack region. Note that the elastic energy of the matrix is reduced, but the increase in the fibre elastic energy is more than enough to offset this reduction.

These energy terms are evaluated by ACK and it can be shown that they are related to each other in a remarkably simple way

$$W_f = \Delta U = V_m \gamma_m \quad (1)$$

where W_f is the frictional energy dissipation per unit area of the composite, and ΔU and $V_m \gamma_m$ are the change in the local elastic energy and the critical matrix crack extension energy release rate per unit area of the composite, respectively, in which V_m and γ_m denote matrix volume fraction and matrix crack extension energy, respectively.

The result given by Equation 1 for a single matrix crack with bridging fibres can be considered as an ideal case of crack deflection due to high-modulus brittle inclusions in a brittle matrix. This result clearly indicates that it is impossible to obtain significant toughening by crack deflection only. The question, now, becomes how much energy can be dissipated by multiple cracking. A reasonable estimate can be given by assuming that the matrix crack spacing is twice the value of the slip length (ACK) which is denoted by w and given by

$$w = \left(a^2 \frac{V_m^2}{V_f} \frac{E_m E_f}{E} \frac{3\gamma_m}{4\tau_s^2}\right)^{1/3} \quad (2)$$

where a is the fibre radius, E_m, E_f, E are Young's moduli of the matrix, fibre and composite, respectively, τ_s is the frictional interface stress, and V_f is the fibre volume fraction.

The energy dissipation before fibre failure per unit volume of the composite, W_d, can be obtained by multiplying the number of cracks, $1/2w$, per unit length with the energy dissipation per unit area of a single crack, to yield

$$W_d = \left[\frac{\gamma_m \tau_s}{a}\right]^{2/3} \left[V_f V_m \frac{4E}{3E_m E_f}\right]^{1/3} \quad (3)$$

Thus, W_d is relatively strongly dependent on the product of the matrix surface energy, γ_m, interface frictional stress, τ_s, and fibre radius, a. Dependence on volume fractions is weaker. However, when $V_f = V_m = 0.5$ the value of W_d is maximized.

The energy dissipation, W_d, which is predicted from Equation 3 is very small compared to the plastic deformation of metals. It can be shown that the strain range of full multiple cracking is at most of the order of the elastic strain range of the composite. Skipping the details, the ratio of the former to the latter is

$$\frac{1}{2} \frac{V_m E_m}{V_f E_f}. \quad (4)$$

3. Energy dissipation after fibre failure, W_p

Section 2 was concerned with matrix failure during which the fibres remain intact. We now consider the effect of fibre failure on the work of fracture for a single crack. For a typical ceramic composite the fibre failure can be described by statistical failure theories of brittle solids. A preliminary discussion of these theories is given followed by an analysis of fibre failure.

The energy dissipation due to fibre pull-out is a function of the length over which the fibres fail within the matrix. This length increases as the applied load is increased. For single crack behaviour the fibre stress in the vicinity of the crack is considerably higher than the rest of the body. Thus, we may neglect the effect of the background stress and assume a triangular axial stress profile with the slope determined by $2\tau_s/a$. The peak fibre stress increases from zero to infinity at constant slope. The fibre failure probability and the average incremental pull-out length are determined from the Weibull analysis. The total work of pull-out is obtained by integrating frictional work per fibre over the whole range of probable lengths as a function of the peak stress.

3.1. Statistical considerations of brittle failure

The tensile strength of a brittle material under uniform loading is governed by the Weibull distribution (see, for example Trustrum and Jayatilaka [5]), with the probability of failure at stress, σ, given by

$$P_f = 1 - \exp\left[-V\left(\frac{\sigma - \sigma_{nf}}{\sigma_0}\right)^m\right] \quad (5)$$

where m is a parameter known as the Weibull modulus that characterizes the flaw distribution, V is the total volume of the material, σ_0 is a normalizing factor and σ_{nf} is the stress below which there is zero probability of failure.

The above expression assumes that there is a single set of volume flaws in the material [6]. If failure occurs due to surface flaws rather than volume flaws, then the total volume, V, in Equation 5 must be replaced by the total surface, S.

The Weibull distribution given by Equation 5 is obtained from the following, more general, expression

$$P_f = 1 - \exp\left[-\iiint_V \left(\frac{\sigma - \sigma_{nf}}{\sigma_0}\right)^m \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right] \quad (6)$$

which provides for the case where the stress distribution in the test specimen is non-uniform.

The Weibull distribution predicts the failure probability of one specimen, but does not indicate where failure occurs within the body. Johnson and Tucker [7] evaluated the volume integral of Equation 6 over a subdomain of a bend specimen to determine the failure probability of the subdomain, provided that the failure probability of the whole body and the stress distribution is known. They show that the probability of failure initiating over a subdomain of the body is proportional to the "effective volume" of the subvolume. Using the same idea, the local probability of failure is governed by

$$R_f(x, y, z) = \frac{[(\sigma - \sigma_{nf})/\sigma_0]^m}{\iiint_V [(\sigma - \sigma_{nf})/\sigma_0]^m \, dV} \quad (7)$$

where R_f denotes the fraction of failed specimens (per unit volume) with origins in the infinitesimal volume, dV. The fraction of the failed specimens with origins in a finite subdomain, V_{sub}, can be obtained by simply integrating R_f over the subvolume. The denominator of Equation 7 is unaffected because it does not depend on (x, y, z). Thus,

$$F_{sub} = \frac{\iiint_{V_{sub}} [(\sigma - \sigma_{nf})/\sigma_0]^m \, dx \, dy \, dz}{\iiint_V [(\sigma - \sigma_{nf})/\sigma_0]^m \, dx' \, dy' \, dz'} \quad (8)$$

where F_{sub} denotes the fraction of the failed specimens with origins in V_{sub}. For a single flaw population the normalization stress, σ_0, is a constant. Therefore, it cancels out in Equations 7 and 8.

The volume integrations in Equations 6 to 8 should be carried out only in the regions where the local stress, σ, exceeds the no-failure stress, σ_{nf}, i.e. $\sigma > \sigma_{nf}$.

As a final note, it should be mentioned that the local stress, σ, in the exponent in Equation 5 would correspond to the maximum tensile principal stress component in the case of multi-axial loading. It is possible to use other brittle failure criteria as an exponent, such as the maximum principal value of the strain tensor, or the dilatational strain energy, etc. In all these cases the constants in Equation 5 must be adjusted appropriately.

3.2. Simple model for fibre failure inside the matrix and pull-out length

The fibres inside the composite will generally tend to fail from surface flaws rather than volume flaws. This is due to the fact that the interface shear stresses enhance the axial stress near the fibre skin and reduce it at the centre [8]. Thus, it may be more proper to use a failure analysis based on surface flaws rather than volume flaws. This can easily be accomplished by replacing total volume, V, in the Weibull distribution (Equation 5) with total surface, S.

The model is shown in Fig. 2. The main matrix crack along which final separation will occur is bridged by fibres which are infinitely long on both sides. The axial stress in the fibre is maximum at the crack surface and drops linearly on both sides, due to the assumption of constant interface shear, τ_s,

$$\sigma(z) = \frac{2\tau_s}{a}(L - z) \quad (9)$$

where $\sigma(z)$ is the mean axial stress in the fibre as a function of the distance, z, from the crack surface, and L is a sampling length along the fibre which is a function of the applied external load. Symmetry allows us to consider the upper half only in Fig. 2. The background fibre stress, $\sigma(L)$, at $z = L$ is assumed to be zero. As the external load is increased the slope of the fibre stress profile remains constant, but the distance L increases in Fig. 2.

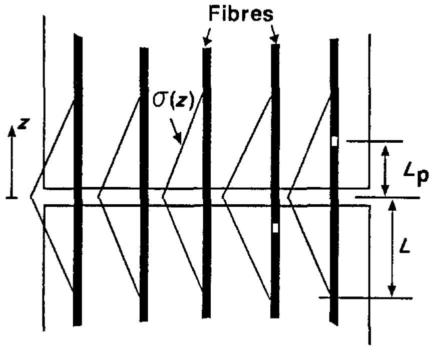

Figure 2 Assumed stress distribution for the bridging fibres and the pull-out length.

In the Appendix, a separate analysis is given where a uniform axial stress is assumed along the length of the fibres.

The fibres may or may not fail under the stress distribution given by Equation 9. The fraction, P_f, of the fibres which have failed is given by the Weibull distribution

$$P_f = 1 - \exp\left[-2\pi a \int_{-L}^{L} \left(\frac{\sigma}{\sigma_0}\right)^m dz\right] \quad (10)$$

where the sampled fibre surface area is $(2\pi aL)$ on either side of the crack surface. The no-failure stress, σ_{nf}, is assumed to be zero.

The probability of fibre failure (Equation 10) can be integrated by using Equation 9,

$$P_f = 1 - \exp\left[-\frac{2\pi}{m+1}\left(\frac{a^2\tau_s^m}{\sigma_0^m}\right)\left(\frac{2L}{a}\right)^{m+1}\right] \quad (11)$$

The unit of σ_0 is nominally (stress $\times$ length$^{2/m}$). This awkward unit can be avoided by introducing another constant into the Weibull formula (Equation 10) which has unit magnitude and dimensions of area. However, this consideration is not important to the present discussion.

For a given fibre with one flaw population, σ_0 is a material constant. When such a fibre is tested under uniform tension, the strength data will depend on the length and radius of the test specimen. Thus,

$$\sigma_0^m = \frac{2\pi a}{\ln 2} H \sigma_f^m \quad (12)$$

where σ_f is the stress which is required to fracture 50% of the fibres with some standard length, H, and radius, a, under uniform tension.

The failure probability given by Equation 11 determines the fraction, P_f, of the fibres which fail prior to the present fibre stress. Under the present stress distribution, an additional fraction, dP_f, fails. Let us determine the average distance from the crack surface at which these additional fibres fail. The pointwise probability of failure is given by Equation 7, which becomes

$$R_f(z) = \frac{m+1}{L}\left(1 - \frac{z}{L}\right)^m \quad (13)$$

in the region where $z < L$. The average failure distance, L_p, see Fig. 2, is therefore,

$$L_p = \int_0^L z R_f \, dz \quad (14)$$

which, using Equation 13 becomes

$$L_p = \frac{L}{m+2} \quad (15)$$

For large values of the Weibull constant, m, the pull-out length L_p, approaches zero. This is expected because a material with large m fails in a deterministic manner at a specific stress level. Thus, the failure occurs at the crack surface where the axial stress is maximum and no pull-out is obtained.

The case where there is a minimum stress, σ_{nf}, below which fibre failure does not occur can be treated by simply replacing σ_f with $\sigma_f - \sigma_{nf}$ and L with $L - L_{nf}$, where L_{nf} is given by

$$L_{nf} = \frac{a\sigma_{nf}}{2\tau_s}. \quad (16)$$

3.3. Energy dissipation, W_p

Energy dissipation per fibre due to pull-out can be obtained by integrating the force due to the frictional wall stress, τ_s, over the slipping distance, L_p. The frictional dissipation is governed by

$$W_{fibre} = \int_0^{L_p} 2\pi a z' \tau_s \, dz' = \pi a \tau_s L_p^2 \quad (17)$$

for a fibre which fails at L_p. The average dissipation per fibre is obtained by integrating Equation 17 over all pull-out lengths, L_p, as a function of failure probability, P_f. The total dissipation per unit area of composite, W_p, is determined by multiplying the average dissipation per fibre with the number of fibres per unit area $V_f/(\pi a^2)$

$$W_p = \frac{V_f \tau_s}{a} \int_0^1 L_p^2 \, dP_f \quad (18)$$

The integration in Equation 18 is carried out by using Equations 11 and 15 to give

$$W_p = \frac{\beta V_f}{4} \frac{a^{(m-3)/(m+1)} \sigma_0^{2m/(m+1)}}{\tau_s^{(m-1)/(m+1)}} \quad (19)$$

where the dimensionless factor, β, is a function of the Weibull constant, m, only

$$\beta(m) = \frac{\Gamma[(m+3)/(m+1)]}{(m+2)^2 \, [2\pi/(m+1)]^{2/(m+1)}} \quad (20a)$$

The gamma function, Γ, is tabulated in handbooks. The values of $\beta(m)$ are plotted in Fig. 3. The following simple formula approximates β within $\pm 5\%$ accuracy over the practical range of m values ($m \geqslant 2$).

$$\beta(m) = \frac{1.1(m-1)}{m(m+2)^2} \quad (20b)$$

For large values of m, $\beta(m)$ and, as a consequence, W_p approach zero.

Let us define an RMS mean length $\langle L_p \rangle$ over which the fibres pull out of the matrix, through

$$\langle L_p \rangle^2 = \int_0^1 L_p^2 \, dP_f \quad (21)$$

This integral has already been evaluated in Equation 18 to determine W_p. Thus, $\langle L_p \rangle$ can be determined from Equations 18 and 19 as

$$\langle L_p \rangle = \frac{\beta^{1/2}}{2} a^{(m-1)/(m+1)} \left(\frac{\sigma_0}{\tau_s}\right)^{m/(m+1)} \quad (22)$$

Alternatively, an arithmetic mean measure can be used and is given by

$$\langle L_p \rangle = \int_0^1 L_p \, dP_f \quad (23)$$

The result is 4% lower than the RMS values obtained from Equation 21 for $m = 3$. For large values of m the difference is very small.

As fibre strength is usually given in terms of an average tensile stress, the pull-out work in Equation 19 will be re-written in terms of the stress, σ_f, which is required to fracture 50% of the fibre specimens of some standard length, H, in uniaxial tension. By using Equation 12,

$$W_p = \frac{\beta}{4} V_f \left(\frac{a}{\tau_s}\right)^{(m-1)/(m+1)} \sigma_f^{2m/(m+1)} (9.065H)^{2/(m+1)} \quad (24)$$

This expression can be related to the critical matrix cracking stress, σ_{cr}, which is given by Budiansky *et al.* [8]. Eliminating (τ_s/a) from Equation 24.

$$W_p \sigma_{cr}^{(3m-3)/(m+1)} = \frac{\beta}{4} \sigma_f^{2m/(m+1)} (9.065H)^{2/(m+1)} \times V_f \left[\frac{6\gamma_m V_f^2 E_f E^2}{V_m E_m^2}\right]^{(m-1)/(m+1)} \quad (25)$$

If the right-hand side of Equation 25 is fixed by specifying all the material constants, then W_p and σ_{cr} can be varied by changing the interface shear τ_s. For typical values of the Weibull constant m, ($m = \sim 5$) the relationship is governed by

$$W_p \sim \frac{\text{const.}}{\sigma_{cr}^2} \quad (26)$$

which shows the inverse interdependence of these quantities.

3.3.1. Numerical example

As an example the glass-ceramic/Nicalon composite, which is examined by Budiansky *et al.* will be examined. In addition to the material data given by these authors we have the following fibre strength properties which are representative of Nicalon (SiC) fibres

$$H = 0.05\,\text{m}$$
$$\sigma_f = 1500 \pm 300\,\text{MPa}, \quad (27)$$
$$m = 5.$$

These data correspond to the case where samples with 5 cm length are tested in uniaxial tension and 50% fail at a stress $\sigma_f = 1500$ MPa.

Using Equation 27 in Equation 19 or 24, the work of pull-out, W_p, for this glass-ceramic/Nicalon composite is

$$W_p = 0.085 \text{ MPa m} = 85 \text{ kJ m}^{-2} \quad (28)$$

which is 1940 × matrix surface energy, γ_m, for LAS-glass. The calculated average pull-out length, $\langle L_p \rangle$ from Equation 22 is

$$\langle L_p \rangle = 0.8 \text{ mm} \quad (29)$$

which is ~100 × fibre radius on either side of the matrix crack.

4. Discussion and conclusions

The work of pull-out, W_p, in Equation 19 or 24, which is determined analytically, is a measure of the composite toughness under uniaxial loading. The actual failure mechanism may or may not be due to a single matrix crack. When failure occurs by activating multiple matrix cracks the triangular stress profile will underestimate the average pull-out length and thus underestimate the work of pull-out. In order to predict the work of pull-out, W_p, for multiple cracking, the triangular stress profile must be replaced by a nearly periodic stress profile with peaks at the matrix cracks. A first-order approximation would be to replace the nearly periodic stress profile with a constant uniform stress as is done in the Appendix.

From a composite material design point of view, both triangular and constant stress profiles show that the work of pull-out depends on the composite material parameters basically in a manner as prescribed by the work of Cooper [1] and Kelly [2] on deterministic aligned discontinuous fibre composites. However, the dependence on the Weibull parameter, m, is different, compare Figs 3 and 4. For a single crack, the fibre strength data should show large scatter with low values of m and for multiple cracking the reverse is true in order to maximize the work of pull-out, W_p.

The work of pull-out, W_p, which is given by Equation 19 or 24 is much higher than the surface energy of the monolithic ceramics as shown by the numerical example. This quantity is considerably enhanced with strong fibres with high values of σ_0. In addition to the fibre strength, the work of pull-out, W_p, also depends on the interface shear, τ_s, fibre radius, a, and the fibre volume fraction, V_f, as can be seen from Equations 19 or 24. These expressions indicate that W_p is nearly proportional to (a/τ_s). In other words, for pull-out purposes it is better to have large diameter fibres which are loosely constrained by the matrix. In this sense, W_p is inversely related to the matrix cracking stress σ_{cr} and to the energy dissipation, W_d, before fibre failure, see Equations 3 and 19. However, it is possible to increase σ_{cr} without changing W_p by lowering the matrix Young's modulus, E_m, and raising the critical matrix crack extension energy, γ_m.

With either the constant or the triangular stress profile, large relative displacements must occur for energy dissipation, see Equations 22 and A5. In general, matrix crumbling occurs which reduces W_p, see the picture of the tensile test specimen after loading which is given by Marshall and Evans [10]. An estimate of the pull-out length which causes matrix crumbling, together with the ultimate tensile strength of the composite are given by Sutcu [3], assuming multiple matrix cracking.

Finally, the energy dissipation W_d before fibre failure which is given by Equation 3 is not a significant quantity. The consequences of this simple result extends beyond the uniaxially reinforced ceramic composites. The result given by Equation 1 implies that the particulate inclusions and crack deflection are not effective toughening mechanisms although they may enhance the modulus and load-bearing capability of the unreinforced matrix. The failure will still be catastrophic.

Acknowledgements

The author thanks Dr William B. Hillig and Dr Curtis A. Johnson at General Electric Research and Development Center for many helpful discussions during the course of this work. Dr W. B. Hillig read the manuscript and made valuable suggestions.

Appendix. The work of pull-out for constant axial fibre stress

Let us assume that the axial fibre stress, σ, is constant along the length of the fibre as opposed to the triangular variation which is sketched in Fig. 2. As the

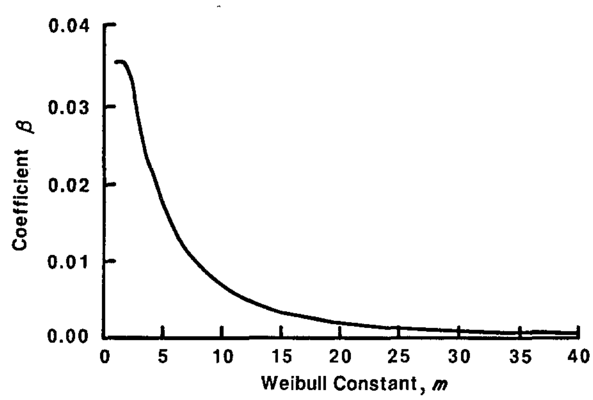

Figure 3 The coefficient β in W_p as a function of the Weibull constant, m (triangular stress profile).

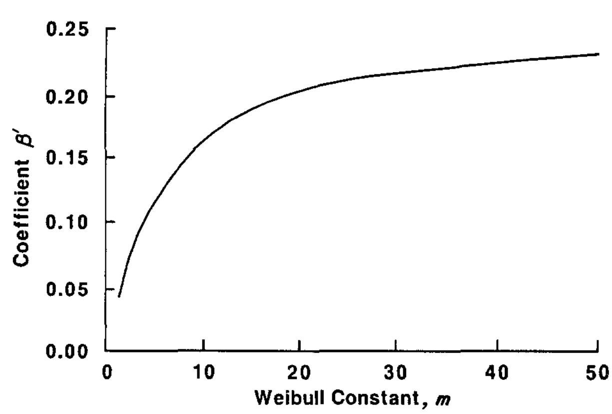

Figure 4 The coefficient β' in W_p for constant stress profile.

external load is increased, the fibre stress increases while maintaining a constant profile. At any given value of σ, it is equally probable to fracture the fibres anywhere along the entire length of the fibre. However, there is a minimum distance, L, from the matrix crack, beyond which the broken fibre ends cannot be pulled out unless σ is increased further. The relationship between L and σ is governed by

$$\sigma = \frac{2\tau_s}{a} L. \tag{A1}$$

The length, L, effectively defines a sampling length along the fibre, which varies as a function of the applied load. The probability of failure, P_f, occurring over the length, L, is governed by the Weibull distribution (Equation 10) which becomes

$$P_f = 1 - \exp\left[-2\pi a^2 \left(\frac{\tau_s}{\sigma_0}\right)^m \left(\frac{2L}{a}\right)^{m+1}\right] \tag{A2}$$

using Equation A1. The fraction of the fibres which fail over the length L have an average failure distance of L_p

$$L_p = L/2 \tag{A3}$$

The work of pull-out, W_p, is given by Equation 18 with L_p and P_f defined by Equation A3 and A2, respectively. Performing the integration the work of pull-out, W_p, for constant stress profile is again governed by Equation 19 with β replaced by β', which is given by

$$\beta'(m) = \frac{\Gamma[(m+3)/(m+1)]}{4(2\pi)^{2/(m+1)}} \tag{A4}$$

The values of β' are plotted in Fig. 4. Comparison with Fig. 3 indicates that the uniform axial stress case predicts a larger work of pull-out, W_p, compared to the triangular case. The pull-out work, W_p, in this case increases with m, as opposed to the previous case, up to an asymptotic value of 1/4.

The average pull-out length for constant stress profile can be obtained from Equation 22 by simply replacing β' with β. If, instead of the RMS mean (Equation 21), the arithmetic mean (Equation 23) is used, then the coefficient $\beta^{1/2}$ in Equation 22 must be replaced by

$$\frac{\Gamma[(m+2)/(m+1)]}{2(2\pi)^{1/(m+1)}} \tag{A5}$$

References

1. G. A. COOPER, *J. Mater. Sci.* **5** (1970) 645.
2. A. KELLY, "Strong Solids", 2nd Edn (Clarendon, Oxford, 1973) pp. 208–11.
3. M. SUTCU, (1987) *Acta Metall.* submitted.
4. J. AVESTON, G. A. COOPER and A. KELLY, "Single and Multiple Fracture", The Properties of Fiber Composites, Conference Proceedings, National Physical Laboratory (IPC Science and Technology Press, Guildford, 1971) pp. 15–26.
5. K. TRUSTRUM and A. S. JAYATILAKA, *J. Mater. Sci.* **14** (1979) 1080.
6. C. A. JOHNSON, Fracture Statistics of Multiple Flaw Distributions, in "Fracture Mechanics of Ceramics", Vol. 5, edited by R. C. Bradt, A. G. Evans, D. P. H. Hasselman and F. F. Lange (Plenum, 1983) pp. 365–86.
7. C. A. JOHNSON and W. T. TUCKER, "Advanced Statistical Concepts of Fracture in Brittle Materials", Proceedings of the Twenty-Third Automotive Technology Development Contractors' Coordination Meeting, 21–24 October, Dearborn, Michigan (1985) pp. 265–70.
8. S. P. TIMOSHENKO and J. N. GOODIER, "Theory of Elasticity", 3rd Edn (McGraw-Hill, New York, 1970) pp. 422–5.
9. B. BUDIANSKY, J. W. HUTCHINSON and A. G. EVANS, *J. Mech. Phys. Solids* **34** (1986) 167.
10. D. B. MARSHALL and A. G. EVANS, *J. Amer. Ceram. Soc.* **68** (1985) 225.

Received 6 April
and accepted 23 June 1987